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SOLUTION OF THREE-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY FOR A PIECEWISE HOMOGENEOUS MEDIUM WITH CONSTANT POISSON'S RATIO

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Singular integral equations of the theory of elasticity are studied for a piecewise homogeneous medium with the same Poisson's ratio. It is shown that a solution can be obtained using the method of successive approximations.

Use of the potential method for the fundamental problems of the theory of elasticity leads to singular integral equations of second kind [1]. In the case of the second internal and external problems, and of the first internal problem, the spectral properties of the integral operators allow the use of the method of successive approximations to obtain a solution.

1. Consider an elastic body D occupying a bounded part of the space R^3, with boundary S_1 which is a Liapunov surface. An inclusion D_2 bounded by a Liapunov surface S_2 $(S_1 \cap S_2 = \varnothing)$ exists within D. We denote the part of D bounded by the boundary $S_1 \cup S_2$ by D_1. Let the Lamé constants of the body D_i be λ_i and μ_i. We assume that the Poisson's ratios are equal to each other, therefore we have $\lambda_1/\lambda_2 = \mu_1/\mu_2 = \beta$. The direction of the normal $\mathbf{n}$ on S_1 pointing away from D_i is assumed positive. The displacement vector of the elastic medium $\mathbf{u}$ and D is a solution of the boundary value problem

$$\begin{gathered}\Delta \mathbf{u} + (1-2\sigma)^{-1} \operatorname{grad} \operatorname{div} \mathbf{u} = 0, \ \mathbf{x} \in D_1, \ D_2 \\ [T_{n1}\mathbf{u}]^+ = \mathbf{f}(\mathbf{x}), \ \mathbf{x} \in S_1; \ \mathbf{u}^+(\mathbf{x}) - \mathbf{u}^-(\mathbf{x}) = \mathbf{r}(\mathbf{x}), \ \mathbf{x} \in S_2 \\ [T_{n2}\mathbf{u}]^+ - [T_{n1}\mathbf{u}]^- = \mathbf{g}(\mathbf{x}), \ \mathbf{x} \in S_2 \\ T_{ni}\mathbf{u} = 2\mu_i \partial\mathbf{u}/\partial n + \lambda_i \mathbf{n} \operatorname{div} \mathbf{u} + \mu_i [\mathbf{n} \cdot \operatorname{rot} \mathbf{u}]\end{gathered} \tag{1.1}$$

Here σ is the Poisson's ratio, $T_{ni}\mathbf{u}$ is the limiting value of the stress operator at the surface with the normal $\mathbf{n}$, and the plus and minus indices indicate whether the limiting value is determined along the positive or negative direction of the normal, respectively.

Since the substitution $\mathbf{u} = \mathbf{u}_1 - \mathbf{v}$, where $\mathbf{v}$ is a solution of the first internal problem for D_2, reduces the problem (1.1) to that with $\mathbf{r}(x) \equiv 0$, we shall assume that $\mathbf{r}(x) \equiv 0$. The functions $\mathbf{f}$ and $\mathbf{g}$ will be regarded as the elements of the Hilbert spaces H_1 and H_2 of vector functions defined on S_1 and S_2, with the scalar product

$$(\varphi_1, \varphi_2)_{H_i} = \int_{S_i} \sum_{j=1}^{3} \varphi_1^{\,j} \varphi_2^{\,j} \, dS$$

Taking tinto account the form of the operator T_{ni}, we reduce the second condition on S_2 in (1.1) to the form

$$[T_{n1}\mathbf{u}]^+ - \beta\, [T_{n1}\mathbf{u}]^- = \beta \mathbf{g}(\mathbf{x}) \tag{1.2}$$

We shall seek the solution in the form of a potential of a simple layer

$$\mathbf{u}(\mathbf{x}) = \int_{S_1} \mathbf{V}(\mathbf{x}, \mathbf{y}) \boldsymbol{\varphi}_1(\mathbf{y}) \, d_y S + \int_{S_2} \mathbf{V}(\mathbf{x}, \mathbf{y}) \boldsymbol{\varphi}_2(\mathbf{y}) \, d_y S \tag{1.3}$$

Here $\mathbf{V}(\mathbf{x}, \mathbf{y})$ is the Kelvin — Somigliano matrix

$$\mathbf{V}(\mathbf{x}, \mathbf{y}) = \{V_{ik}(\mathbf{x}, \mathbf{y})\}_{i, k=1}^{3}$$

$$V_{ik}(x, y) = (16\pi\mu_1(1 - \sigma))^{-1}[(3 - 4\sigma)\delta_{ik} / |\mathbf{x} - \mathbf{y}| + (y_i - x_i)(y_k - x_k) / |\mathbf{x} - \mathbf{y}|^3]$$

The function (1.3) satisfies the Lamé differential equations in $D_1 \cup D_2$ and fulfils, by virtue of the continuity of the potential of a simple layer, the first condition on S_2 in (1.1). The second condition on S_2 (condition (1.2)) leads to the equation

$$\boldsymbol{\varphi}_2(\mathbf{x}) + \alpha \int_{S_2} T_{n1}\mathbf{V}(\mathbf{x}, \mathbf{y}) \boldsymbol{\varphi}_2(\mathbf{y}) \, d_y S + \alpha \int_{S_1} T_{n1}\mathbf{V}(\mathbf{x}, \mathbf{y}) \boldsymbol{\varphi}_1(y) \, d_y S = \frac{1 - \alpha}{2} \mathbf{g}(\mathbf{x}) \tag{1.4}$$

where $\alpha = (1 - \beta) / (1 + \beta)$. For the principal contact problem in which the elastic body fills the whole space, the above equation was given in [1] where the convergence of the method of successive approximations for (1.4) was also shown.

The equation obtained from the condition on S_1 has the form

$$\boldsymbol{\varphi}_1(x) + \int_{S_1} T_{n1}\mathbf{V}(\mathbf{x}, \mathbf{y}) \boldsymbol{\varphi}_1(\mathbf{y}) \, d_y S + \int_{S_2} T_{n1}\mathbf{V}(\mathbf{x}, \mathbf{y}) \boldsymbol{\varphi}_2(\mathbf{y}) \, d_n S = \mathbf{f}(\mathbf{x}) \tag{1.5}$$

Let us consider the system (1.4), (1.5) of integral equations, introducing the following notation:

$$K_{ji}\boldsymbol{\varphi} = \int_{S_i} T_{n1}\mathbf{V}(\mathbf{x}, \mathbf{y}) \boldsymbol{\varphi}(y) \, d_y S, \quad \mathbf{x} \in S_j$$

$$K_{ji}^*\boldsymbol{\varphi} = \int_{S_j} [T_{n1}\mathbf{V}(\mathbf{x}, \mathbf{y})]' \boldsymbol{\varphi}(\mathbf{y}) \, d_n S, \quad \mathbf{x} \in S_i; \quad i, j = 1, 2$$

$$\mathbf{T}_\alpha = \{(T_\alpha)_{ij}\}, \quad (T_\alpha)_{1j} = K_{1j}, \quad (T_\alpha)_{2j} = \alpha K_{2j}, \quad j = 1, 2$$

$$\mathbf{T}_\alpha^* = \{(T_\alpha^*)_{ij}\}, \quad (T_\alpha^*)_{i1} = K_{1i}^*, \quad (T_\alpha^*)_{i2} = \alpha K_{2i}^*$$

$$\mathbf{T}_{-1} \equiv \mathbf{T}; \quad \boldsymbol{\varphi} = \mathrm{col}(\boldsymbol{\varphi}_1; \boldsymbol{\varphi}_2), \quad \mathbf{F}_0 = \mathrm{col}(\mathbf{f}; (1 - \alpha)\mathbf{g}/2)$$

Here a prime denotes transposition and interchange of arguments, $\mathrm{col}(\psi_1, \psi_2)$ is a vector assuming the value ψ_i on the surface S_i, and $\mathbf{T}_\alpha^*$ is an operator conjugate to $\mathbf{T}_\alpha$ in H. The operators $K_{ii} : H_i \to H_i$ are singular integral operators and $\Sigma(K_{ii}) \subseteq [-1, 1]$ [1]. (here and henceforth $\Sigma(A)$ denotes the spectral set of the operator A). The operators K_{ij} $(i \neq j)$ are completely continuous from H_j into H_i.

Now we can write the system (1.4), (1.5) in the form

$$(I + \mathbf{T}_\alpha)\boldsymbol{\varphi} = \mathbf{F}_0 \tag{1.6}$$

where I is the identity operator. We note that when $\alpha = -1$ $(\mathbf{T}_\alpha = \mathbf{T})$, the system (1.6) corresponds to the second fundamental problem for a doubly connected body bounded by the surface $S = S_1 \cup S_2$ [1].

2. Let us study the equation (1.6). Since the spectral radius K_{22} is equal to unity $(\rho(K_{22}) = 1)$, the operator $I + \alpha(K_{22})$ has a bounded inverse on H_2 for any $\alpha \in (-1, 1)$. Writing now φ_2 from (1.4) in terms of φ_1 and substituting it into (1.5), we obtain

$$\varphi_1 + K_{11}\varphi_1 + R\varphi_1 = \mathbf{f} + (I + \alpha K_{22})^{-1}(1-\alpha)/2 \qquad (2.1)$$

Here R is an operator, containing K_{12} and K_{21} as factors and therefore fully continuous. Since the operator $I + K_{11} : H_1 \to H_1$ is a Noetherian operator with zero index, it follows that the Fredholm alternative holds for (2.1) as well as for (1.6), the latter holding by virtue of the equivalence of the transformation which has been carried out (the boundedness of $(I + \alpha K_{22})^{-1}$).

Let $\Psi(\mathbf{x}) = \mathbf{a} + [\mathbf{b}\cdot\mathbf{x}]$ ($\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ be sets of arbitrary constants and let square brackets denote a vector product) be the rigid displacement vector defined for all $\mathbf{x} \in R^3$; $\psi_i(x) = \Psi(\mathbf{x})$ for $\mathbf{x} \in S_i$ is the trace of $\Psi(\mathbf{x})$ on the surface S_i.

Consider the vector $\Psi_\alpha(\mathbf{x}) = \mathrm{col}\,((1-\alpha)\psi_1/2; \psi_2)$. Let $\Psi_{\alpha j}$ ($j = 1, 2, \ldots 6$) be linearly independent functions of Ψ_α constructed with the help of six, linearly independent vector constants $\mathbf{a}, \mathbf{b}$. Direct substitution confirms that $\{\Psi_{\alpha j}\}_{j=1,2,\ldots 6} \subset N(I + \mathbf{T}_\alpha^*)$ and $\{\mathrm{col}\,(0; \psi_{2j})\}_{j=1,2,\ldots 6} \subset N(\alpha I + \mathbf{T}_\alpha^*)$ for $\alpha \in [-1, 1]$ (here and henceforth $\{\Phi_i\}_{i=1,2,\ldots k}$ denotes the linear envelope of the vectors Φ_i, $N(A)$ is the subspace of zeroes of the operator A). We shall show that $\{\mathrm{col}\,(0; \psi_{2j})\}_{j=1,2,\ldots 6} = N(-I + \mathbf{T}^*)$. Indeed, the equation $(-I + \mathbf{T})\varphi = \mathbf{F}$ is equivalent to the second fundamental boundary value problem of the theory of elasticity (for a body consisting of two parts lying, respectively, within S_2 and outside S_1) which has a solution if and only if $\mathbf{F} \in \{\mathrm{col}\,(0; \Psi_{2j})\}^\perp_{j=1,2,\ldots 6}$. Consequently we have $\{\mathrm{col}\,(0; \Psi_{2j})\}^\perp_{j=1,2,\ldots 6} = (-I + \mathbf{T})H = N(-I + \mathbf{T}^*)^\perp$.

Consider the space $H^\circ = \{\varphi \in H : (\psi_{1j}, \varphi_1)_{H_1} = (\psi_{2j}\,\varphi_2)_{H_2} = 0 \quad j = 1, 2 \ldots 6\}$. Direct substitution shows that for $\alpha = 1$

$$H^\circ = \{\mathrm{col}\,(0; \psi_{2j})\}^\perp_{j=1,2,\ldots 6} \cap \{\Psi_{\alpha j}\}^\perp_{j=1,2,\ldots 6}; \quad \mathbf{T}_\alpha H^\circ \subset H^\circ$$

Let us consider the contraction T_α° of the operator T_α onto H°. We shall show that $\Sigma(\mathbf{T}^\circ) \not\ni -1, 1$. We recall that the points $-1, 1 \in \Sigma(\mathbf{T})$ are isolated points in $\Sigma(\mathbf{T}) \subset [-1, 1]$ [1]. Let $\varphi \in N(I + \mathbf{T}^\circ)$ and $\varphi \in H^\circ$. Then $\varphi \in N(I + \mathbf{T}^\circ) \cap N(I + \mathbf{T}^*)^\perp$. Since the pole -1 of the resolvent of the operator $\mathbf{T}$ is simple [1], it follows that $\varphi \equiv 0$. Let us now assume that $-\varphi + \mathbf{T}\varphi = 0$ and $\varphi \in H^\circ$. Then we have simultaneously $\varphi \in N(-I + \mathbf{T})$ and $\varphi \in N(-I + \mathbf{T}^*)^\perp$ and $\varphi = 0$ by virtue of the fact that the pole 1 of the resolvent of the operator $\mathbf{T}$ is simple, i.e. $1 \not\in \Sigma(\mathbf{T}^\circ)$.

Thus $\Sigma(\mathbf{T}^\circ) \subset [-1 + \delta, 1 - \delta]$ for some $\delta > 0$, and this implies that $\rho(\mathbf{T}^\circ) < 1$, i.e. a norm exists, equivalent to the initial norm, in which $\|\mathbf{T}^\circ\|_* = q < 1$.

At this stage we note that for any $\mathbf{F} = \mathrm{col}\,(\mathbf{F}_1; \mathbf{F}_2) \in H$ we have

$$\|\mathbf{T}_\alpha \mathbf{F}\|_H \leqslant \|K_{11}\mathbf{F}_1 + K_{12}\mathbf{F}_2\|_{H_1} + |\alpha|\,\|K_{22}\mathbf{F}_2 + K_{21}\mathbf{F}_1\|_{H_2} \leqslant \|\mathbf{TF}\|_H$$

From this, by virtue of $\mathbf{T}_\alpha H^\circ \subset H^\circ$ we have for any $\mathbf{F} \in H^\circ$ and any integral $n > 0$,

$$\|\mathbf{T}_\alpha{}^n\mathbf{F}\| \leqslant \|\mathbf{T}\mathbf{T}_\alpha^{n-1}\mathbf{F}\| \leqslant \ldots \leqslant \|\mathbf{T}^n\mathbf{F}\| \leqslant c\|\mathbf{T}^n\mathbf{F}\|_* \leqslant cq^n\|\mathbf{F}\|_*$$

where c is a constant entering the condition of equivalence of the norms. From this it follows, that the series

$$\sum_{k=0}^{\infty}(-T_\alpha)^k F$$

converges on the norm for any $\mathbf{F} \in H^\circ$. Let $\mathbf{F} \in \{\Psi_{\alpha j}\}_{j=1,2,\ldots 6}^{\perp}$. Since $(\alpha I + \mathbf{T}_\alpha)\mathbf{F} \in N(\alpha I + \mathbf{T}_\alpha{}^*)^\perp \subset \{\text{col}\,(0; \psi_{2j})\}_{j=1,2,\ldots 6}^{\perp}$, then for $\alpha \neq 1$ $(\alpha I + \mathbf{T}_\alpha)\mathbf{F} \in H^\circ$ and the series

$$\sum_{k=0}^{\infty}(-T_\alpha)^k(\alpha I + \mathbf{T}_\alpha)\mathbf{F}$$

converges. We can now confirm that when $\alpha \in [-1, 1)$, the function

$$\varphi = \frac{1}{1-\alpha}\left[\mathbf{F} - \sum_{k=0}^{\infty}(-\mathbf{T}_\alpha)^k(\alpha I + \mathbf{T}_\alpha)\,\mathbf{F}_0\right] \qquad (2.2)$$

satisfies the equation (1.6). Thus when $\mathbf{F}_0 \in \{\Psi_{\alpha j}\}_{j=1,2,\ldots 6}^{\perp}$, a solution of (1.6) exists. This implies that we have, by virtue of the Noetherian character of the operator $I + \mathbf{T}_\alpha$, $\{\Psi_{\alpha j}\}_{j=1,2,\ldots 6} = N(I + \mathbf{T}_\alpha{}^*)$.

The sufficiency of the condition $(\mathbf{f}, \psi_{1j}) + (\mathbf{g}, \psi_{2j})$ $(j = 1, 2, \ldots, 6)$ (which is equivalent to $\mathbf{F} \in \{\Psi_{\alpha j}\}_{j=1,2,\ldots 6}^{\perp}$) for the solvability of the initial boundary value problem, has been shown in [1]. The iterative process

$$\varphi_{(0)} = (\alpha I + \mathbf{T}_\alpha)\,\mathbf{F}, \quad \varphi_{(k)} = -\mathbf{T}_\alpha\varphi_{(k-1)}, \quad k = 1, 2, \ldots$$

yields the series appearing in (2.1), and the singular integrals appearing in $T_\alpha\varphi_{(k)}$ can be computed at each step using the procedure given in [2]. The results obtained can be applied to other fundamental problems for a piecewise homogeneous medium with the same value of the Poisson's ratio.

By assuming in the formulation of the problem that $V(\mathbf{x} - \mathbf{y}) = 1 / |\mathbf{x} - \mathbf{y}|$ (u, φ_1, φ_2, f, g are scalar functions), we arrive at the Neumann problem for the Laplace's operator for a composite region with a given gradient discontinuity at the boundary. The results also hold in this case.

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